BELIEFS AND PROBABILITIES: THE ERRORS THAT REMAIN ARE MINE ALONE

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ABSTRACT

Imagine that the preface to a professor's book implicitly asserts that all the propositions in the rest of his or her book are true, but explicitly acknowledges that experience would suggest some errors remain among those propositions. The professor thereby seems paradoxically to believe inconsistent statements. But, in fact, this famous preface paradox is an illusion. The first statement is a belief reflecting epistemic uncertainty, while the second is a probabilistic statement about aleatory uncertainty. If one were to convert the probability into a belief, one would see that the author rationally holds perfectly consistent beliefs.

Likewise, the lottery paradox is resolved. Remarkably, the resolution of these philosophy paradoxes sheds important light on legal evidence and proof: once one realizes that legal factfinding deals in beliefs, not probabilities, many of the law's proof paradoxes vaporize. All those paradoxes reveal a generally applicable and powerful principle of rational thought: if, in the presence of epistemic uncertainty, a person believes fact x and believes fact y because each passes the threshold for belief, then the person believes x AND y together. The explanation lies in the fact that epistemic uncertainty calls for nonadditive logic, which employs the MIN rule for conjunction rather than the product rule. The significance is broad, as it maps where one can logically believe a string of beliefs as a narrative chain.

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I. INTRODUCTION

Professor David Makinson formulated the preface paradox in 1965,[1] and it has been debated ever since. It hypothesizes an author who implicitly affirms that he or she believes all the complex propositions declared in the rest of the book, but who goes on to acknowledge in the book's preface (or an article's star footnote) that broad experience would suggest some errors remain among those propositions. This hypothesis is not a flight of imagination, as the paradox would arise whenever an author writes something like "the remaining errors are mine alone."

In other words, the author, merely by writing the book, has asserted the truth of all its propositions, s_1 *AND* . . . *AND* s_n. But the author's preface also seems to have asserted the conjunction's negation, ~(s_1 *AND* . . . *AND* s_n), because he or she thinks a nonspecific one or more of the propositions are false. The experience that produced this feeling of fallibility lowers the other beliefs but not below the threshold level for belief. The author thus seems to make inconsistent statements irrationally. Accordingly, we would not be justified in accepting anything asserted in the book.

Lest it be thought that something was lost in my translation, here is how Makinson set forth the paradox, while he nevertheless defended the author's impetus:

> The author who writes and believes each of s_1, . . . , s_n and yet in a preface asserts and believes ~(s_1 & . . . & s_n) is, it appears, behaving very rationally. Yet clearly he is holding logically incompatible beliefs: he believes each of s_1, . . . , s_n, ~(s_1 & . . . & s_n), which form an inconsistent set. The man is being rational though inconsistent. More than this: he is being rational even though he believes each of a certain collection of statements, which *he knows* are logically incompatible.[2]

There's the rub: the author's two inconsistent statements sound rather rational together. Those who want to escape the paradox thus undertake the task of explaining how logic could allow both (s_1, . . . , s_n) and ~(s_1 & . . . & s_n) to be true.

To begin, the paradox arises from the intuitive principle of conjunctive closure of beliefs: "If S is rational, then if S believes *A* and S believes *B*, then S believes *A and B*."[3] So, if the author believes s_1 to s_n, then the author should believe their conjunction—but does not!

To rescue the coexistence of the author's two assertions, Makinson

1. D.C. Makinson, *The Paradox of the Preface*, 25 ANALYSIS 205 (1965).
2. *Id.* at 205.
3. Simon J. Evnine, *Believing Conjunctions*, 118 SYNTHESE 201, 201 (1999).

simply jettisoned the principle of conjunctive closure. His idea was that absent this principle, the string of beliefs says nothing about belief in the conjunction.[4] In the final two paragraphs of his article, Makinson convolutedly asserts, without justification, that believing a series of propositions does not mean that you believe their conjunction.[5] Subsequent theorists have tried to justify his move.[6] One of them offered three arguments, none of which is especially satisfying:

> First, it is surely absurd to credit someone with beliefs of things of which he can have no understanding. . . . But a person who can only "hold a few ideas in his head" might severally believe each of a series of propositions, where the number of propositions is so large that he is unable to understand or consider their conjunction. . . . Second, a person who believes that p and q, may have considered whether p and considered whether q, without having considered whether p *and* q, in which case he does not *consciously* believe that p and q. . . . Third, while each of a number of statements may be highly probable, relative to the evidence, their conjunction may be highly improbable relative to it.[7]

Admittedly, the principle of conjunctive closure appears vulnerable at least to that third argument, as the principle seems to ignore probability's product rule. For example, "the probability that p *and* q is 2/3 x 2/3, i.e. 4/9, i.e. *less* probable than not, which justifies a belief that ~(p & q)."[8]

However, jettisoning the principle of conjunctive closure has significant epistemic costs.[9] Accordingly, other theorists accept that principle in general and instead argue, with mighty struggles and nondefinitive reasons, that the author does not really disbelieve the conjunction and, therefore, no contradiction exists.[10] The commonality among these paths is to argue that, even if the principle is not always valid, a disbelief in the conjunction does not arise where the principle is valid. First, the easiest path switches the discussion from beliefs to probabilities, creating from that third argument a

4. *See* Makinson, *supra* note 1, at 207.

5. *See id.*

6. *See, e.g.*, Henry E. Kyburg, Jr., *Conjunctivitis*, *in* INDUCTION, ACCEPTANCE, AND RATIONAL BELIEF 55, 77 (Marshall Swain ed., 1970) ("[I]t seems preposterous to suppose that all of our inductive knowledge has to be embodiable in a single fat statement.").

7. John N. Williams, *The Preface Paradox Dissolved*, 53 THEORIA 121, 127, 129, 131 (1987).

8. *Id*. at 132.

9. *See* Hannes Leitgeb, *The Review Paradox: On the Diachronic Costs of Not Closing Rational Belief Under Conjunction*, 48 NOÛS 781, 792 (2014) ("Either the traditional epistemology of rational belief preserves closure under conjunction, or it has a more serious problem than it is normally thought to have.").

10. *See, e.g.*, Evnine, *supra* note 3, at 220–23 (utilizing the idea of sub-systems of beliefs); *cf.* John L. Pollock, *The Paradox of the Preface*, 53 PHIL. SCI. 246 (1986) (developing a difficult theory of nomic probability to dismantle the paradox).

limited exception when dealing with probability-based beliefs.[11] The author's "beliefs" $s_1, \ldots, s_n$ are supposedly that each proposition is probably or very probably correct, and the product rule does not necessarily imply that their conjunction is probably incorrect. The result is a principle of "restricted conjunctive closure": if $s_1, \ldots, s_n$ are believed, then their conjunction is believed, *unless* its probability falls below the threshold level for belief.[12] Second, a more complicated but similar path would reject the Lockean thesis,[13] so as to argue that conjunctive closure is rational for categorical (or flat-out or qualitative) beliefs but not for partial (or gradated or quantitative) beliefs.[14] So, a principle of "very restricted conjunctive closure" would say this: if $s_1, \ldots, s_n$ are believed, then their conjunction is believed, *unless* any of the string of beliefs is a partial belief.[15] Third, the broadest path to upholding conjunctive closure takes the discussion from probabilities fully back to beliefs.[16] This route's premise is that the beliefs in $s_1, \ldots, s_n$ are a string of categorical beliefs, while the negation of the conjunction $\sim(s_1 \& \ldots \& s_n)$ is not a categorical belief and, instead, is merely a partial belief. Thus, the two statements are talking about such different things that they cannot be inconsistent.

I, too, maintain that the author's two statements are consistent, but I do so on grounds that are both easier to grasp and more complete. Without rejecting the principle of conjunctive closure or the Lockean thesis, I maintain that, given epistemic uncertainty, one can believe both $(s_1, \ldots, s_n)$ and $\sim(s_1 \& \ldots \& s_n)$ without encountering a contradiction. The reason is that, as they are stated in the paradox, the former is a string of beliefs and the latter is a probability.

To explain my reasoning, I must define probabilities and beliefs before analyzing the paradox. Why is this preface paradox worth the trouble of analyzing? Because the preface problem raises the basic question of when you can logically believe a string of beliefs as a narrative chain.

11. *See, e.g.*, James Hawthorne & Luc Bovens, *The Preface, the Lottery, and the Logic of Belief*, 108 MIND 241 (1999); Christopher New, *A Note on the Paradox of the Preface*, 28 PHIL. Q. 341 (1978).

12. *See* Gerhard Schurz, *Rational Belief in Lottery- and Preface-Situations: Impossibility Results and Possible Solutions*, *in* LOTTERIES, KNOWLEDGE, AND RATIONAL BELIEF: ESSAYS ON THE LOTTERY PARADOX 128, 140 (Igor Douven ed., 2021).

13. The Lockean thesis is that beliefs will arise when confidence exceeds some threshold degree of confidence that the agent deems sufficient for belief. *See* James Hawthorne, *The Lockean Thesis and the Logic of Belief*, *in* DEGREES OF BELIEF 49, 49 (Franz Huber & Christoph Schmidt-Petri eds., 2009); Schurz, *supra* note 12, at 130.

14. *See* H.L. HO, A PHILOSOPHY OF EVIDENCE LAW: JUSTICE IN THE SEARCH FOR TRUTH 124–29 (2008) (discussing categorical and partial beliefs); Keith Frankish, *Partial Belief and Flat-Out Belief*, *in* DEGREES OF BELIEF, *supra* note 13, at 75.

15. *See* Schurz, *supra* note 12, at 129–30.

16. *See* JONATHAN E. ADLER, BELIEF'S OWN ETHICS 198–209 (2002).

II. PROBABILITIES

The preface paradox involves making statements about truth in the face of uncertainty. Two kinds of uncertainty are in play here: aleatory and epistemic.

Aleatory Uncertainty. Aleatory uncertainty comes from any process that appears random to us, like flipping coins under ideal conditions. By its nature, this randomness is a built-in part of existence. It, along with any free will, is what keeps us, at present at least, from seeing the universe as wholly deterministic. Obviously, randomness even-handedly shows no patterns because patterns would mean that there is more nonrandom information extricable from the data. It is irreducible, in the sense that we think accessible information would not remove the randomness. Of course, our view of what is random has shrunk over the centuries. Aleatory uncertainty expresses what is inherently unknowable under our current understanding of the universe. Given that current understanding, we would still be left with aleatory uncertainty even if everything conceivably accessible were known.

Epistemic Uncertainty. Epistemic uncertainty comes from the ignorance left by incomplete, inconclusive, ambiguous, dissonant, or unreliable evidence; from the indeterminacy produced by the vagueness of our concepts and our expressed perceptions of the real world; and from the limits on our imaginative powers as to what we do not know. While aleatory uncertainty captures random unsureness by expressing a first-order estimate of which way things will randomly turn out, these three sources of epistemic uncertainty create a second-order unsureness about that estimate.

Traditional Probability. Our predominant way of handling uncertainty is traditional probability, be it a classical, frequentist, or subjective probability system.[17] Such a system builds on Kolmogorov's tripartite axiomatization: nonnegativity (there are no negative probabilities), normalization (a sure event has a probability of 1), and additivity (the disjoint probability of mutually exclusive events equals the sum of their separate probabilities, so that the measure of an event's happening and the measure of its not happening added to 1). Those probabilities run on a scale from 0 to 1 (or 0% to 100%), indicating how probable the event will resolve in the affirmative. Traditional probability can act as a supplement to bivalent logic. That widely prevailing logic assumes all propositions are true or false, allocating all propositions to true (one) or false (zero) and excluding the middle. A probability thus can give the odds of truth, p, with additivity's necessary implication being that the odds of falsity are $1-p$.

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17. *See generally* Alan Hájek, *Interpretations of Probability*, *in* STANFORD ENCYCLOPEDIA OF PHILOSOPHY (Edward N. Zalta ed., 2019), https://stanford.io/3e9Jfm6 (mapping the whole range of probability theory).

An example might help to illuminate how traditional probability handles uncertainty in factfinding:

> Assume that the evidence on fact *a*'s existence, say, Dave's identity as the culprit, remains weak after hearing all the available, but very imperfect, evidence. Nonetheless, the affirmative evidence forcefully outweighs the evidence that Dave was not the culprit. If you were forced to bet on whether accepting the identification is the right course and you wanted to use odds to express the preceding sentence, then perhaps you would say Prob(*a*) = 80% and Prob(*not-a*) = 20%. That is, even though your feeling of belief one way or the other is fairly weak, if you had to allocate all your belief to either yes or no, you would formulate odds of 80/20.

Is the 80% figure, alone, an accurate representation of your state of mind as to the existence of *a*? No. An 80% chance on airtight evidence would feel very different from this 80% chance on thin evidence. That is, traditional probability disregards the presence of epistemic uncertainty. It assumes your knowing all there is to know that is nonrandom. When you state a probability, you are acting as if you know all that there is to know, and you are leaving as uncertain only the acceptedly unknowable effects of randomness in fixing the yes-or-no outcome.

To recapitulate, the world contains (1) known information, (2) unknown but conceivably knowable information, including both known unknowns and unknown unknowns, which generate epistemic uncertainty, and (3) random unknowables. Traditional probability looks to the known information of type (1) to create odds that express type (3)'s random effects. However, it mishandles type (2)'s epistemic uncertainty by treating it as randomness. The essential premise of additive probability is assuming that the decisionmaker knows everything conceivably knowable, thus allowing the treatment of all uncertainty as random variation. Type (2) is not actually random, however, so this simplification is a source of error. This simplification is sometimes appropriate, but not when the decisionmaker needs to keep track of epistemic uncertainties in order to combine information accurately.

An additional observation, essential for my purposes, concerns the rules for combining propositions in a traditional probability system. These rules derive from the axiom of additivity. For independent propositions *a* and *b*, the product rule says that the probability of the conjoined propositions is the product of their probabilities. For interdependent propositions, the probability operation for conjunction is Prob(*a*) multiplied by the conditional Prob(*b*|*a*), so it is still multiplicative. These two rules collectively are called the general product rule or just the product rule. As to disjoining, De Morgan's rule provides that the disjunction of two independent statements (say, each .25 probable) equals the negation of the product of the negations of those statements ($1-.75^2 = .44$). For interdependent statements *a* and *b*, the product-rule portion of De Morgan's rule would involve Prob(*a*) and Prob(*b*|*a*).

III. BELIEFS

Nonadditive multivalent logic, such as fuzzy logic or possibility theory, provides a comprehensible way of accounting for epistemic uncertainty.[18] This widely accepted and highly developed logic rejects, as assumptions, the law of the excluded middle and the axiom of additivity. The omitted assumptions make this logic more general than bivalent logic and traditional probability, which appear as special cases of multivalent logic.

Nonadditive multivalent logic looks to the known information of type (1) in order to create degrees of belief that are subject to type (3)'s random effects. But it can record type (2)'s epistemic uncertainty as uncommitted belief.

Degrees of Belief. "Belief," for my purposes, is a justified mental acceptance of a statement as true.[19] A belief must rest on the person's attempt to express the state of the real world, as represented by the evidence assembled by reasonable means and processed logically. To qualify as a belief, the person's conviction may be absolute or certain. Alternatively, the conviction need only satisfy the standard of proof fixed as the threshold for beliefs. The threshold might require the belief to exceed its complement that represents represents possible disbelief. Or perhaps the threshold might require only that the belief exceed the actual disbelief.

A "degree of belief," or Bel(*a*) or multivalent belief, is the belief holder's estimate of the extent, on a scale from 0 to 1, to which proposition *a* has been fully proved.[20] Importantly, the complement of Bel(*a*) is the degree to which *a* has not been proved; it is not the smaller degree to which *not-a* has been proved. Thus, a degree of belief can coexist with a degree of disbelief, or Bel(*not-a*). Moreover, the belief holder can withhold part of its full belief, leaving belief uncommitted to a degree that depends on the quality and quantity of the evidence and the presence of other unknowns. This uncommitted belief represents epistemic uncertainty, and it causes the belief holder's degrees of belief and disbelief to add to less than one. Degrees of belief and disbelief are, therefore, nonadditive and nonprobabilistic.

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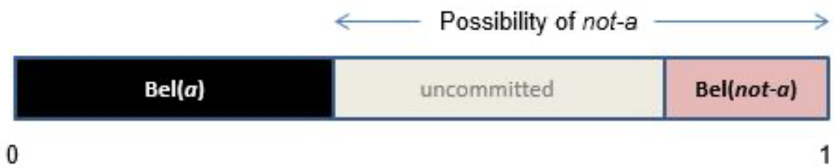
18. *See generally* KEVIN M. CLERMONT, A GENERAL THEORY OF EVIDENCE AND PROOF: FORMING BELIEFS IN TRUTH (forthcoming 2023).

19. *See* Jessica Moss & Whitney Schwab, *The Birth of Belief*, 57 J. HIST. PHIL. 1 (2019).

20. *See* L. JONATHAN COHEN, THE PROBABLE AND THE PROVABLE (1977).

These degrees of belief and disbelief can be represented by a so-called belief function:[21]



If the factual issue is whether Dave was the culprit, the inquiry starts with the whole range of belief standing as uncommitted. At this initial point, everything is indeterminate because the current lack of evidence requires the factfinder to withhold all belief. The proper representation of lack of proof is zero belief in the affirmative position—but also zero disbelief. The uncommitted belief is the entirety or 1, meaning that *a* is completely possible, as is *not-a*. Belief function theory thus utilizes the very useful notion of lack of belief, as distinguished from disbelief.

As evidence comes in, some of the factfinder's uncommitted belief should start to convert into a degree of belief in *a*'s existence, and the proof will usually have the effect of generating an active degree of belief in its nonexistence. These degrees of beliefs in *a* and *not-a* constitute partial truths. The zone between Bel(*a*) and Bel(*not-a*) represents the remaining uncommitted belief. When we say, after processing the evidence, that Bel(*a*) = 0.40, we are not saying that Bel(*not-a*) = 0.60. We are saying only that the proof is such that to a degree of 0.60, which could represent uncommitted belief in part or in whole, *a* has not been fully proven to be true, and *not-a* is possible. This statement differs from a probability of 40% that *a* would somehow be revealed as completely true, and 60% that it would be revealed as false.

Conjunction. Bivalent logic can handle only the values of 1 and 0. But degrees of belief can take values in between. Therefore, to handle these multiple values, we need to use the operators of nonadditive multivalent logic. For conjunction and disjunction, the operators are the MIN and MAX rules, respectively. So, for example, the conjunction of Bel(*a*) and Bel(*b*) is the minimum of the two degrees of belief. This operator is a more general replacement for the product rule, which appears as a special case in an additive system. Moreover, the same MIN rule applies whether the degrees of belief are independent or interdependent, unlike the product rule.[22]

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21. *See* GLENN SHAFER, A MATHEMATICAL THEORY OF EVIDENCE (1976).

22. *See* Richard Bellman & Magnus Giertz, *On the Analytic Formalism of the Theory of Fuzzy Sets*, 5 INFO. SCI. 149, 151–55 (1973).

Why do beliefs and probabilities combine differently? The reason is that a degree of belief alone does not tell you anything about a disbelief. It just tells you the degree to which the affirmative of the fact was fully proved. So, when you combine a belief in *a* with a belief in *b*, your conjunctive belief is proved as far as the lesser degree of the two beliefs. By contrast, probabilistic odds of truth in an additive system also tell you the odds of falsity. When you conjoin the odds of *a* and *b*, you must calculate both the conjunctive odds of truth and the disjunctive odds of falsity. Let me explain.

As we formulate degrees of belief subject to epistemic uncertainty, they combine into a chain as strong as its weakest link. No way exists to multiply the beliefs, and if there were a way, it would make no intuitive sense. Conjunctive closure prevails instead. Indeed, here lies the source of and justification for the principle of conjunctive closure. The principle holds true by virtue of the MIN rule in a nonadditive system: if you believe *a* and you believe *b*, then you believe *a AND b*.

However, the odds are a different measure. Odds of *a* dictate the odds of *not-a*. Conjoin *a* with *b* as probabilities, and the disjunctive odds of *not-a* or *not-b* rise. There is no reservoir of epistemic uncertainty to buffer the effect of conjunction. As the disjunctive odds rise, the conjunctive odds of uncertain *a* and *b* drop below both *a*'s and *b*'s odds. Therefore, in an additive system, the principle of conjunctive closure cedes to the product rule. In short, it makes good sense to employ different operators for multivalent beliefs and probabilistic odds.

Broader Comparison to Probability. Recall the other differences between beliefs and probabilities. A degree of belief is the fractional share of one's full belief that one attributes to a proposition's truth. These multivalent beliefs consider epistemic uncertainty as well as aleatory uncertainty. By contrast, traditional probability reports the truth in terms of the approximate number of times the proposition would be revealed as true if you were to repeat the scenario 100 times—or what odds would make truth and falsity equally attractive to you as someone betting on revelation. This measure of probability buries all epistemic uncertainty and reflects only aleatory uncertainty. Beliefs and probabilities, therefore, report different measures appropriate in different circumstances.

Nonetheless, beliefs are not radically different from probabilities. Nonadditive multivalent logic is the more general logic, and it can reduce to bivalent logic and traditional probability if one re-assumes the law of the excluded middle and, hence, the axiom of additivity. Thus, probabilities can be stated as degrees of belief, and degrees of belief can be transformed into probabilities.

First, converting a probability into a degree of belief is easy in the case of a pure probability that describes a situation with no epistemic uncertainty. Multivalent logic can effortlessly express the probability as the membership in a set. That is, it can treat the probability as the degree to which the imagined universe of all tests of the proposition would belong to the set of positive results.[23] The probability can then be stated as a belief function, where the degree of belief equals p and the degree of disbelief equals $1-p$. However, many probabilities will entail hidden epistemic uncertainty. To convert these into degrees of belief, epistemic uncertainty must be uncovered and expressed. That is, the underlying evidence for the proposition must be connected to the proposition by a series of permissible but uncertain inferences; the evidence may have to be discounted for defects in credibility; the degree of belief may have to be adjusted to account for the probative value of the absence of other proof; and so on. The result will be a belief function with uncommitted belief.

Second, how do people convert belief functions into probabilistic odds of having uncovered the truth, if it were to be somehow revealed? They would perform one of the variety of theorist-proposed transforms, which can convert the credal (from the Latin for belief) framework into the pignistic (from the Latin for betting) framework.[24] Rather than leaving some belief uncommitted, they would thereby allocate all their belief between the two possible outcomes of true and false, committing more belief to "true" and more disbelief to "false" and retaining no measure of epistemic uncertainty about the allocation. This transform process might involve the nature of the involved epistemic uncertainty and might invoke attitudes toward risk and loss, helping to make the proper transform method quite contestable and complicated. The bigger point is this: you can bet with very little information but you nevertheless must allocate all belief between the two possible outcomes to place your bet.

The reason to use multivalent beliefs in a task like factfinding is that they deliver more accurate results whenever the thinker encounters epistemic uncertainty. When thinking people need to keep track of epistemic uncertainty for accuracy's sake, they should employ multivalent beliefs. There is, indeed, ground to think that people do reason with multivalent logic in the face of epistemic uncertainty.[25]

IV. RESOLUTION

The resolution of the preface paradox lies in realizing that the professorial author's two statements are different in kind, the first being a belief involving epistemic uncertainty and the second being a probability

23. *See* BART KOSKO, FUZZY THINKING: THE NEW SCIENCE OF FUZZY LOGIC 55−64 (1993).

24. *See* Barry R. Cobb & Prakash P. Shenoy, *A Comparison of Methods for Transforming Belief Function Models to Probability Models*, *in* SYMBOLIC AND QUANTITATIVE APPROACHES TO REASONING WITH UNCERTAINTY 255 (Thomas Dyhre Nielsen & Nevin Lianwen Zhang eds., 2003).

25. *See* Jane Friedman, *Suspended Judgment*, 162 PHIL. STUD. 165 (2013).

involving only aleatory uncertainty.

First, the author believes that each of the book's propositions is true. Presumably, the author has generated a series of arguable propositions subject to disagreement. That is the common situation for nonfiction authors, and the preface's other statement about some remaining errors largely confirms that this common situation prevails. Thus, the author is saying that taking all arguments into account produced a belief in the truth of each proposition. In doing so, the author retained some belief as uncommitted to reflect epistemic uncertainty. The author also formed a degree of disbelief in each proposition, which equated to the disjunction (or maximum) of the degrees of belief in all positions contrary to the particular proposition. The author is asserting at the least that, for each proposition, the degree of belief in its truth exceeded a threshold for belief. Then, by the MIN rule, the conjunction of degrees of belief in the set of propositions equals the minimum of those degrees of belief. The author believed the weakest proposition in the book to be true. And so, he or she believed the whole chain of propositions.

Second, the author also thinks that some errors survived. This statement, however, does not rest on a disjunction of the degrees of disbelief in each proposition. Instead, it is a calculation of the odds of some proposition being wrong. In other words, the author is saying, "I'd bet there are some mistakes. I base this not on the strength of my propositions, but on my view that random human fallibility endangers any proposition." For an analogy, if there were ten independent propositions with *probabilities* of 90%, then by De Morgan's rule, the probability of at least one of them being wrong would be 65%. This is a calculation of odds, which ignores epistemic uncertainty. It is not a belief.

Calculating the belief inherent in the 65% probability of error would require accounting for epistemic uncertainty. The degree of belief in the existence of some error would be the disjunction of the degrees of *disbelief* in each proposition (Bel(*not*-s_1 *OR* . . . *OR* *not*-s_n), or Bel(*disj*)). By the MAX rule, this value is rather low, being not high enough to exceed the conjunction of the degrees of *belief* in the book's propositions registering as true (or Bel(*conj*)). The minimum belief exceeds the maximum disbelief. None of the disbeliefs is strong enough to be accepted as true. Consequently, awash in epistemic uncertainty, the author does not believe there is an error.

The pudding's proof indeed comes by asking whether the author would bet on whether there is an error somewhere. Let us say that the author has a considerable Bel(*conj*) and a smallish Bel(*disj*), as any author should when committing to writing a book. If the author's book contains only a few propositions, the smart bet would be against there being an error. But as the number of propositions increases, that bet becomes less attractive. Why? The author wants to predict something about the whole sequence of revelation on each proposition. The author might transform each of $s_1, \ldots,$

s_n into a probability, thereby generating a string of nonnegligible chances of error. Then, De Morgan's rule would produce from those chances a sizable Prob(*disj*). In accordance with additivity, this result implies a decreased Prob(*conj*). Consequently, some authors would end up betting there probably is some error, even if they believe in their book.

How can these two views of the world coexist? Easily, because they rest on different assumptions. In writing the book, the author accepts the world as it is, with all its epistemic uncertainty. This world calls for a measure of how far proven each proposition is, a degree of belief whose complement expresses both epistemic uncertainty and disbelief. Such a set of multivalent beliefs employs the MIN rule for conjunction of beliefs. Meanwhile, the betting author must switch assumptions to a bivalent world subject only to randomness. Each proposition is either true or false. In a sequence of propositions, the chance of some error popping up seems to increase continually in accordance with De Morgan's rule. Obviously, calculations based on the assumptions of different worlds will produce different conclusions. They might look contradictory, even if they are perfectly consistent in reality.

What is the lesson here? There are two logical systems for handling uncertainty, each based on its own assumptions. One should employ the system that best represents the world one is contemplating. If one is working in a complex world full of epistemic uncertainty, then use multivalent beliefs. Publish that book. If one is contemplating a sequence of yes/no revelations, then use traditional probability. Place your bet.

Multivalent beliefs are a sound basis for action in the face of epistemic uncertainty. On the one hand, beliefs comprise the decisionmaker's best representation of the world as proven by the evidence. Deciding in conformity with the belief structure would feel comfortable. Notably, that decision would be the more accurate one and, thus, would lead to error-cost minimization. On the other hand, using odds instead would require the decisionmaker to assume that the truth or falsity of each proposition will somehow be revealed. The decisionmaker might next transform each belief function into a probability, and then use the product rule to calculate the odds of the propositions' conjunction. This process would introduce a big source of error. The reason is that the transform involves converting all epistemic uncertainty into randomness. This is a false step. Epistemic uncertainty is an unknown. There is no proof that the epistemically uncertain zone leans one way or the other, toward belief or disbelief. But treating it as somehow randomly allocated to both belief and disbelief magnifies disbelief, which depresses the affirmative belief in the conjunction when the product rule is applied. And that representation is a misrepresentation of the decisionmaker's best view of the world. The transform is necessary to predict revelation, but such a prediction is an unnecessary step for deciding in accordance with the most accurate view of the facts.

In other words, one should use beliefs when one needs to act upon unresolved epistemic uncertainty. If one wants to act in the face of persistent epistemic uncertainty, one should act in accordance with the facts that meet the threshold level of belief under the MIN rule. But would it ever be smart for the author to bet there is a mistake? Yes, it could be. Common sense suggests that in a book of a thousand (or a million) propositions, an error would creep in. The multitude itself is evidence of a mistake, evidence that was not considered in setting the beliefs in $s_1, \ldots, s_n$.[26] This new probabilistic evidence feeds into the construction of the multivalent belief and corresponding disbelief in the propositions' conjunction. Thus, a new belief function could arise with the belief in the propositions' conjunction falling below the disbelief. This new belief function is the best basis for the transformation into odds for betting.

In sum, the resolution of the preface paradox comes down to the fact that beliefs and probabilities are different animals which are not readily comparable. Beliefs express the extent to which propositions are proved, while probabilities express likelihood in frequentist terms. The author's first statement is a justified belief in truth, and the second is an abstract statement about randomness. Given a set of propositions, the author could rationally state a belief that all the propositions are true while also stating that the odds favor at least one being wrong. The two statements are talking about different things and, as such, are not contradictory.

V. DISCUSSION

The principle of conjunctive closure survives the preface paradox nicely, but only for those nonadditive beliefs. It does not supplant the product rule for additive probabilities. Thus, a principle of "slightly restricted conjunctive closure" avoids contradiction in that preface and elsewhere:

> If $s_1, \ldots, s_n$ are believed, then their conjunction is believed, *unless* some of these propositions constitute additive probabilities that cause the conjunction to fall below the threshold level for belief.

I can say "slightly restricted" because in the real world, epistemic uncertainty normally prevails. In that case, contradiction can be avoided without rejecting the principle of conjunctive closure or the Lockean thesis, and without making any fancy steps like relying on multivalent logic's

26. *Cf.* Robert Hoffman, *Mr. Makinson's Paradox*, 77 MIND 122 (1968) (contending that the evidence for the string of beliefs could be independent of the evidence for disbelieving the conjunction). *But cf.* New, *supra* note 11, at 343 (refuting Hoffman's use of that contention).

comfort with contradiction. Conjunctive closure remains part of rational thought.

This reasoning that resolves the preface paradox also resolves the lottery paradox. It is similar to the preface paradox but was formulated a little earlier.[27] It imagines a fair lottery that sells a thousand tickets and will have one winning ticket. It is reasonable to believe that any particular ticket will not win, and so by conjunctive closure it appears reasonable to believe that no ticket will win. But, paradoxically, you know that some ticket will win. Again, this paradox depends on confusing beliefs with probabilities. The "belief" that any particular ticket will lose does not operate as a belief. Here, no epistemically uncertain beliefs are involved at all, and hence applying conjunctive closure would create an error. There are only probabilities, calling for the calculus of probability. Each ticket has .001 chance of winning. By Kolmogorov's additivity axiom, the tickets together have a winning chance of a thousand .001s added together, or 1.000. And by Kolmogorov's normalization axiom, 1 is the probability of the sure event that some ticket will win. The paradox springs from applying conjunctive closure to believe that no ticket will win. *The principle of conjunctive closure does not apply to pure probabilities.*

Moreover, this reasoning has broad practical use. In law, there is the conjunction paradox. It arises from the rule that to prove a case, the proponent must prove a series of essential facts, or elements, such as "the culprit was Dave" and "the culprit was negligent"; meanwhile, fairness and efficiency depend on whether the plaintiff's tale of liability is more likely than all the narratives of nonliability put together; nevertheless, a civil plaintiff is not required to prove that liability is more likely than not, but only that each element of the claim meets the standard of proof.[28] Paradoxically, under the product rule, each element's being sufficiently probable does not guarantee that their conjunction's probability will meet the standard of proof.[29] Once one realizes that legal proof deals in beliefs and not probabilities, however, one will put aside the product rule and instead wheel out the MIN rule.[30] Then, if each element satisfies the standard of proof, their conjunction should too. *The principle of conjunctive*

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27. HENRY E. KYBURG, JR., PROBABILITY AND THE LOGIC OF RATIONAL BELIEF 197 (1961); *see* LOTTERIES, KNOWLEDGE, AND RATIONAL BELIEF: ESSAYS ON THE LOTTERY PARADOX (Igor Douven ed., 2021).

28. *See* 3 KEVIN F. O'MALLEY, JAY E. GRENIG & WILLIAM C. LEE, FEDERAL JURY PRACTICE AND INSTRUCTIONS: CIVIL § 104.01 (6th ed. 2011) ("Plaintiff has the burden in a civil action, such as this, to prove every essential element of plaintiff's claim by a preponderance of the evidence. If plaintiff should fail to establish any essential element of plaintiff's claim by a preponderance of the evidence, you should find for defendant as to that claim.").

29. *See* Charles Nesson, *The Evidence or the Event? On Judicial Proof and the Acceptability of Verdicts*, 98 HARV. L. REV. 1357, 1385–90 (1985).

30. *See* KEVIN M. CLERMONT, STANDARDS OF DECISION IN LAW: PSYCHOLOGICAL AND LOGICAL BASES FOR THE STANDARD OF PROOF, HERE AND ABROAD 145–220 (2013).

closure applies to multivalent beliefs.

Many other paradoxes would benefit from rethinking along similar lines. Self-referential paradoxes, such as "This sentence is false," provide an illustration.[31] This liar paradox is significant because it exposes a defect in our prevailing language and logic.[32] Turning to multivalent logic but taking the extra step of drawing on its comfort with contradiction, theorists have found a solution to self-reference, or at least a circumvention: dialetheism, which holds that a sentence can be both true and false and which thereby implies the rejection of bivalence. Dialetheism here builds on Kripke's theory of truth, which invoked a three-valued logic.[33] This turn to multivalence is more of a circumvention than a solution because, even if it dissolves the paradox, it raises other profound questions about truth. Solving the preface paradox involves separating apples from oranges, namely, probabilities from beliefs as the two different ways a person can describe the world. But circumventing the liar paradox poses the question of whether truth is ultimately divisible in reality, in the way that belief in truth is divisible.

More directly and much more generally, the preface-paradox reasoning justifies inferential reasoning in all its ubiquity. Take a string of epistemically uncertain inferences leading from E to F to G to a conclusion, C. Each necessary step must be believed. The belief in F is a conditional belief dependent on E. Let $\Rightarrow$ mean "implies." Theorists have derived an inference rule that says if α and $(\alpha \Rightarrow \beta)$ are proved as beliefs to degree λ and μ, respectively, then we can assert β at degree MIN(λ, μ).[34] So, if one views E as a belief and views the inference (E $\Rightarrow$ F) as a conditional belief, then the belief in F will be the minimum of those two beliefs. As one goes up a chain of inferences, one will believe the latest inference to the degree of the minimum of all the preceding beliefs in the chain. So, when an inference rests on inferences, the conjoined strength of belief drops to the likelihood of the least likely inference. The conjoined belief in the conclusion is as strong as the weakest inference. That is, one will believe C to the extent of the weakest link in the chain.

The big point here is that conjunctive closure is critical for reasoning in huge swaths of life—in law, science, daily existence, and so on. Most

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31. *See generally* Thomas Bolander, *Self-Reference*, *in* STANFORD ENCYCLOPEDIA OF PHILOSOPHY (Edward N. Zalta ed., 2017), https://plato.stanford.edu/entries/self-reference/ ("If we assume the sentence to be true, then what it states must be the case, that is, it cannot be true. If, on the other hand, we assume it not to be true, then what it states is actually the case, and thus it must be true. In either case we are led to a contradiction.").

32. *See* ALFRED TARSKI, *The Concept of Truth in Formalised Languages*, *in* LOGICS, SEMANTICS, METAMATHEMATICS 152 (1956) (translation of 1935 paper).

33. *See* Saul Kripke, *Outline of a Theory of Truth,* 72 J. PHIL. 690 (1975).

34. *See* GIANGIACOMO GERLA, FUZZY LOGIC: MATHEMATICAL TOOLS FOR APPROXIMATE REASONING 113–16 (2001).

problems require conjunction under epistemic uncertainty. That task calls for nonadditive logic, which employs the MIN rule for conjunction rather than the product rule. So, we make sense of and navigate through the world with the essential aid of the principle of conjunctive closure. Jettisoning the principle unnecessarily to resolve the preface paradox would lead us to cognitive catastrophe:

> Our "information society" harbors a number of contexts such as handbooks or lexica of all sorts, collecting high numbers of facts each being well confirmed though not absolutely certain. It seems doubtful that one should really withdraw one's belief in each of their well-established facts just because this is the only way of upholding the rationality conditions . . . .[35]

VI. CONCLUSION

The logical fact is that the traditional probability of truth and the multivalent belief in truth are different measures.

- Traditional probability leads to all sorts of logical problems because of its ignoring epistemic uncertainty.
- Multivalence allows a full accounting for uncertainty, although the result is a nonadditive system.
- Conjunction of nonadditive multivalent beliefs, given their definition as the degree to which the proposition has been fully proved, leads inevitably to the MIN rule: when you combine a belief in *a* with a belief in *b*, your conjunctive belief is proved as far as the lesser degree of the two beliefs.

Hence, the mathematical fact is that degrees of belief follow the MIN and MAX rules, not probability's product rule.

The key insight is that there are two major tools for deciding under uncertainty: beliefs and probabilities. The paradoxes of the preface, lottery, and conjunction deal in confusion between beliefs and probabilities. Focusing on which measure is in play, and comparing either beliefs to beliefs or probabilities to probabilities, will cause the paradoxes to vaporize. As a matter of rational thought, the bottom line is that we should embrace the following principle: **if $s_1, \ldots, s_n$ are believed, then their conjunction is believed, *unless* we are dealing with some additive probabilities that reduce the conjunction below the threshold level for belief.**

35. *See* Schurz, *supra* note 12, at 139.